International Journal of Theoretical Physics, Vol. 25, No. 8, 1986

Superposition and Quantum Mechanics

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Received December 10, 1985

This work is primarily concerned with finding those statements or observations from which quantum mechanics can reasonably be said to follow. Within the context of characterizing quantum mechanics as any probability field (with bounded probability density) whose associated stochastic velocity field is governed by a differential equation of first order in time, it is shown that the single statement required is the stipulation that the superposition principle is satisfied. This is demonstrated by showing that only the Schrödinger equation is an acceptable dynamic description for such probability fields if the superposition principle is to hold.

1. INTRODUCTION

The present work is primarily concerned with characterizing the quantization process; that is, with finding those few physical assumptions which imply quantum theory as we customarily know it. Of course, there already exist several developments whose purpose is to somehow characterize quantum theory. For instance, we have Dirac's (1947) prescription for replacing classical Poisson brackets with operator commutators, or von Neumann's (1955) method of associating quantum operators with classical quantities once the operators associated with position and momentum are assumed. There are other techniques as well, such as those of Weyl (1950), Rivier (1957), and Yvon (1948). In all these cases, however, we do not have what one could call a demonstration of quantum mechanics following as a consequence of certain physical assumptions or observations. In fact, these approaches in no way give any compelling reason in principle for believing the Schrödinger equation, nor do they provide a unified description of the Schrödinger equation together with the statistical assertions of quantum mechanics. In all this, however, there is an exceptional approach which does yield the Schrödinger equation and the statistical assertions from a

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0020-7748/86/0800-0829$05.00/0

unified framework. It is the path integral approach developed by Feynman (1965). However, in this case we must accept as given the expression for the contribution to the kernel coming from each constituent classical path; and this expression follows in no obvious way from more basic observations or assumptions.

What we are seeking in this work is a succinct set of physical statements from which quantum mechanics reasonably follows. Within the context of our characterization of quantum mechanics as any probability field (with bounded probability density) whose associated stochastic velocity field is governed by a differential equation of first order in time, we feel that we have found the single statement needed; it is the requirement that the equations of the theory satisfy the superposition principle.

In the following presentation this conclusion will be developed in stages. In Section 2, we make precise the notion of superposition and, in fact, also discuss linear superposition and what we shall call general linear superposition as well. The nontrivial result that stochastic classical mechanics (considered as an inviscid Eulerian probability fluid) does not satisfy the superposition principle will also be proven here, as a preliminary to later considerations.

In Section 3 it is shown that any probability field that has a probability density that is bounded and with a stochastic velocity field (defined by the continuity equation) that is irrotational and is governed by a differential equation of first order in time can only satisfy the superposition principle if the dynamics is, in fact, governed by the Schrödinger equation.

In Section 4 there is a discussion in which the very lengthy preceding development of Section 3 is summarized and brought into focus, so that one can see just what has been accomplished.

Finally, in Section 5, it is shown that any probability field of the above kind, except one whose stochastic velocity field is not irrotational, cannot be expected to satisfy the superposition principle.

The chief conclusion of all this, then, is that only the Schrödinger equation is an acceptable dynamic description for probability fields with bounded probability densities, and with stochastic velocity fields governed by differential equations of first order in time, if the superposition principle is mandated. We also add the obvious, which is that once the Schrödinger equation is established, so, then, is Feynman's weighting factor and his "derivation" of the statistical aspects of the theory as well.

2. SUPERPOSITION AND CLASSICAL MECHANICS

In this section we first define what it means for a system of differential equations to satisfy a superposition principle, linear superposition principle, and general linear superposition principle. It will then be demonstrated that a common stochastic version of classical mechanics does not satisfy the superposition principle.

2.1. Definitions

Using a nomenclature that will be relevant later, suppose that we have a system of two partial differential equations in the dependent variables (fields) $r(\mathbf{x}, t)$ and $\varphi(\mathbf{x}, t)$, where $\mathbf{x}$ and t signify position and time, respectively. The equations may involve spatial as well as time differentiations, of arbitrary order and degree. We say that the system satisfies the *superposition principle* if, for every pair of solutions of the system $r_1(\mathbf{x}, t)$, $\varphi_1(\mathbf{x}, t)$ and $r_2(\mathbf{x}, t)$, $\varphi_2(\mathbf{x}, t)$, there is another solution $r(\mathbf{x}, t)$, $\varphi(\mathbf{x}, t)$, where $\varphi = \varphi(r_1, r_2, \varphi_1, \varphi_2)$ and $r = r(r_1, r_2, \varphi_1, \varphi_2)$, and where these functions are independent of the nature of the functions $r_1(\mathbf{x}, t)$, $r_2(\mathbf{x}, t)$, $\varphi_1(\mathbf{x}, t)$, and $\varphi_2(\mathbf{x}, t)$. [We call the functions $\varphi(r_1, r_2, \varphi_1, \varphi_2)$ and $r(r_1, r_2, \varphi_1, \varphi_2)$ *universal.*]

Further, we say that the system satisfies a *linear superposition principle* in case there exist variables $\eta = \eta(r, \varphi)$ and $\xi = \xi(r, \varphi)$ (in terms of which the system of equations can be expressed) which are universal functions of r and φ such that if r_1, φ_1 and r_2, φ_2, and hence r, φ, are solutions, then (η_1, ξ_1) and (η_2, ξ_2) are solutions and $\eta = \alpha\eta_1 + \beta\eta_2$, $\xi = \alpha\xi_1 + \beta\xi_2$ is also a solution, where $\eta_i \equiv \eta(r_i, \varphi_i)$ and $\xi_i \equiv \xi(r_i, \varphi_i)$ for $i = 1, 2$, and α and β are arbitrary, real constants. In particular, we see that if the system satisfies the linear superposition principle, and if (η, ξ) is a solution, then so is $(\alpha\eta, \alpha\xi)$ for any real constant α.

Finally, we say that the system satisfies the *general linear superposition principle* in case the system satisfies the linear superposition principle and also satisfies the property that, whenever (η, ξ) is a solution, so is $(\alpha\eta + \beta\xi, \alpha\xi - \beta\eta)$ for any real constants α and β.

Some comments about these definitions are in order. We note that if the system satisfies the general linear superposition principle, then it also satisfies the other two superposition principles. Further, if the system satisfies the linear superposition principle, then it satisfies the superposition principle. Conventional quantum theory, as we shall elaborate later, satisfies the general linear superposition principle; classical mechanics satisfies none of them, and we shall find another mechanics that satisfies the linear but not the general linear superposition principle.

2.2. Classical Mechanics

We now consider classical mechanics as a stochastic theory and prove that it does not satisfy the superposition principle. This result, which is interesting in itself, will be used to frame the context and formalism for a later, more general discussion relevant to quantum theory.

We consider an ensemble of noninteracting classical systems, each system being subject to the same external field $V(\mathbf{x})$. For simplicity let each system be composed of a single particle. We make the usual continuity assumptions so that we may speak of the hydrodynamic-like (stochastic) velocity $\mathbf{v}(\mathbf{x}, t)$ of this "fluid" in the abstract configuration space of the ensemble. Assuming that $\mathbf{v}$ is irrotational, so that we may put $\mathbf{v} = m^{-1}\nabla\varphi$ (where m is the mass of the particle) for some scalar function φ, we assume that the ensemble is described by the dynamic equation

$$m\frac{d\mathbf{v}}{dt} = -\nabla V \tag{1}$$

as well as the continuity equation

$$\nabla \cdot (r^2\nabla\varphi) + mr^2_{,t} = 0 \tag{2}$$

where $r^2(\mathbf{x}, t)$ is the probability density of the replicas in configuration space. We note that equation (1) is merely Newton's law for a single particle, since in the present case $\mathbf{v}$ is both the single-particle velocity and the stochastic velocity field, which assures conservation of system number via equation (2).

Using the relation $\mathbf{v} = m^{-1}\nabla\varphi$, we can reexpress (1) as

$$\frac{(\nabla\varphi)^2}{2m} + V = -\varphi_{,t} \tag{3}$$

where the form here (but not its content) is the same as the Hamilton–Jacobi equation of classical mechanics. Equations (2) and (3) comprise our description of stochastic classical mechanics (which is the same as that of an Eulerian fluid).

We are interested in investigating these two equations in relation to the superposition principle because this provides us a way of introducing an approach which will be very useful later when discussing the same problem for the equations of quantum mechanics, to which the above equations are very similar.

We assume then that the superposition principle holds for this system and investigate some of the consequences. We shall find that this leads to contradictions, forcing the conclusion that the superposition principle cannot hold for this system after all.

Consider equations (2) and (3) with any two solutions (r_1, φ_1), (r_2, φ_2). Let $\varphi = \varphi(r_1, r_2, \varphi_1, \varphi_2)$, $r = r(r_2, r_2, \varphi_1, \varphi_2)$ be another solution, where φ and r are universal functions. Then we can write

$$\nabla\varphi = \alpha_i\nabla\varphi_i + \beta_i\nabla r_i; \qquad \varphi_{,t} = \alpha_i\varphi_{i,t} + \beta_i r_{i,t} \tag{4}$$

and

$$\nabla r = A_i \nabla \varphi_i + B_i \nabla r_i; \qquad r_{,t} = A_i \varphi_{i,t} + B_i r_{i,t} \tag{5}$$

where the summation convention is used throughout, the summation being from 1 to 2, and $\alpha_i = \partial\varphi/\partial\varphi_i, \ldots,$ etc.

Substituting these expressions into equations (2) and (3) gives

$$\nabla \varphi_i \cdot \nabla \varphi_j \alpha_i \alpha_j + \nabla r_i \cdot \nabla r_j \beta_i \beta_j + 2\nabla \varphi_i \cdot \nabla r_j \alpha_i \beta_j + 2mV = -2m(\alpha_i \varphi_{i,t} + \beta_i r_{i,t}) \tag{6}$$

and

$$\begin{aligned} &r(\alpha_i \nabla^2 \varphi_i + \beta_i \nabla^2 r_i + \nabla \alpha_i \cdot \nabla \varphi_i + \nabla \beta_i \cdot \nabla r_i) \\ &\quad + 2(\alpha_i \nabla \varphi_i + \beta_i \nabla r_i) \cdot (A_i \nabla \varphi_i + B_i \nabla r_i) \\ &\quad + 2m(A_i \varphi_{i,t} + B_i r_{i,t}) = 0 \end{aligned} \tag{7}$$

Now, in equation (6), which is all we shall need for present considerations, we reexpress $\varphi_{i,t}$ and $r_{i,t}$ on the right-hand side in terms of spatial derivatives via equations (2) and (3), giving, after collecting terms,

$$\begin{aligned} &\nabla \varphi_i \cdot \nabla \varphi_j \{\alpha_i \alpha_j - \alpha_i \delta_{ij}\} + \nabla r_i \cdot \nabla r_j \beta_i \beta_j \\ &\quad + \nabla \varphi_i \cdot \nabla r_j \{2\alpha_i \beta_j - 2\beta_i \delta_{ij}\} - \beta_i r_i \nabla^2 \varphi_i + 2mV(1 - \alpha_1 - \alpha_2) = 0 \end{aligned} \tag{8}$$

As a consequence of superposition, this equation must hold for all functions r_i and φ_i; also, the α_i and β_i must be universal functions of their arguments.

Now, choosing φ_1, φ_2, r_1, and r_2 to all be constants (at a given t), we find that

$$\alpha_1 + \alpha_2 = 1 \tag{9}$$

when $V \neq 0$. But since the α_i are universal functions, this relation must be valid in general (i.e., even when $V \equiv 0$).

Rewriting equation (9) as

$$\frac{\partial \varphi}{\partial \varphi_1} = 1 - \frac{\partial \varphi}{\partial \varphi_2} \tag{10}$$

we see that

$$\varphi = \varphi_1 + f(r_1, r_2, \varphi_1, \varphi_2) \tag{11}$$

for some function f (this very weak statement will lead to one of content presently).

Therefore, we have that

$$\frac{\partial f}{\partial \varphi_1} + \frac{\partial f}{\partial \varphi_2} = 0 \tag{12}$$

which, in turn, implies that

$$f = f(r_1, r_2, \varphi_2 - \varphi_1) \tag{13}$$

yielding the relation $\varphi = \varphi_1 + f(r_1, r_2, \varphi_2 - \varphi_1)$.

Again returning to equation (8) and this time choosing r_1, r_2, and φ_2 as constants and $\nabla\varphi_1$ as a constant vector (all at a given t), we get the relation

$$\alpha_1^2 = \alpha_1 \tag{14}$$

from which we conclude that $\alpha_1 = 0$ or 1. In the same way, choosing r_1, r_2, and φ_1 as constants and $\nabla\varphi_2$ as a constant vector yields the result that $\alpha_2 = 0$ or 1. From equation (9) this means that either $\alpha_1 = 1$, $\alpha_2 = 0$ or $\alpha_1 = 0$, $\alpha_2 = 1$. Now, $\alpha_2 = 0$ implies that $f = f(r_1, r_2)$, which implies that $\varphi = \varphi_1 + f(r_1, r_2)$. But by symmetry we must also have $\varphi = \varphi_2 + g(r_1, r_2)$ for some function g, which then implies that $\varphi_2 - \varphi_1 = f(r_1, r_2) - g(r_1, r_2)$. But this cannot be, since φ_1 and φ_2 are independent of r_1 and r_2. Choosing $\alpha_1 = 0$ leads again to the same difficulty.

So we see that the stochastic form of classical mechanics considered does not satisfy the superposition principle.[2]

3. SUPERPOSITION AND CONVENTIONAL QUANTUM THEORY

In this section we consider the constraint placed on any conceivable quantum theory by the superposition principle as defined earlier. More specifically, we shall prove that the only quantum theory possible, whose stochastic velocity field (to be defined shortly) is irrotational and is governed by a differential equation of first order in time, has a bounded probability density, and satisfies the superposition principle, is that described by the customary Schrödinger equation. In this connection, we describe the quantum theory as *conventional* if its stochastic velocity field is irrotational.

We begin by constructing the form of the most general conventional quantum theory possible. In the usual way we envisage a set of noninteracting replicated systems (each consisting of just one point particle in an external field, for simplicity) forming our sample space. Let $r^2(\mathbf{x}, t)$ denote the observed density of measured positions at time t (assuming no prior measurements). Then we (nonuniquely) *define* the *stochastic velocity* $\mathbf{v}(\mathbf{x}, t)$ by the requirement that, for given $r^2(\mathbf{x}, t)$, $\mathbf{v}$ assures conservation of the number of systems. That is, $\mathbf{v}$ satisfies the continuity equation

$$\nabla \cdot (r^2\mathbf{v}) + r^2_{,t} = 0 \tag{15}$$

for all $\mathbf{x}$, t.

[2]It is also known, of course, that customary Newtonian mechanics does not satisfy the superposition principle. But this fact does not seem to be simply related to the result in this paper for stochastic classical mechanics.

For any given $r^2(\mathbf{x}, t)$ it is easy to show that there exists such a $\mathbf{v}$ (in fact, infinitely many).[3] We furthermore assume that $\mathbf{v}$ is irrotational, and we write

$$\mathbf{v} = \frac{1}{m} \nabla \varphi \tag{16}$$

where m is the mass of the one-particle system being considered, and $\varphi(\mathbf{x}, t)$ is some scalar function.

Now, we assume that

$$m \frac{d\mathbf{v}}{dt} = \mathbf{B} \tag{17}$$

where $\mathbf{B}$ is a vector field somehow depending on the system state (see below) under consideration (but not depending on time derivatives of the state).

In general, we may write

$$\mathbf{B} = \nabla \Omega + \nabla \times \mathbf{A} \tag{18}$$

for some scalar function Ω and vector field $\mathbf{A}$. We expect that for each system state [i.e., specification of $\varphi(\mathbf{x}, t)$ and $r^2(\mathbf{x}, t)$]Ω and $\mathbf{A}$ will be different.

From the irrotationality of $\mathbf{v}$, equation (17) then yields the relation

$$\nabla \left[\partial_t \varphi + \frac{1}{2m} (\nabla \varphi)^2 \right] = \nabla \Omega + \nabla \times \mathbf{A} \tag{19}$$

And if the fields are such that they all vanish at infinity (say, for certain r^2 and φ) we can then conclude (for such states) that

$$\partial_t \varphi + \frac{1}{2m} (\nabla \varphi)^2 = \Omega \qquad \text{and} \qquad \nabla \times \mathbf{A} = 0 \tag{20}$$

Here, we certainly expect that Ω is a scalar function of the system state, i.e., of r, φ, and their spatial derivatives. If this were not so, then the equation would have different forms for different states, which possibility we reject. Moreover, Ω must be invariant under rotations and coordinate inversions; this is obviously so when $V = 0$ and, in order for the equations to always have the same form even in external fields, must also be so when $V \neq 0$. Also, since the above equation is to have the same form for all possible

[3]For example, the expression

$$\mathbf{v}(\mathbf{x}, t) = \frac{1}{4\pi r^2} \nabla \int \frac{r^2_{,t}\, d\mathbf{x}'}{|\mathbf{x}' - \mathbf{x}|}$$

is easily shown to satisfy the continuity relation for any specified $r^2(\mathbf{x}, t)$. Further, we may add on any solenoidal field/r^2 to the right-hand side here and still have a suitable $\mathbf{v}$.

states (whether or not the fields vanish at infinity), the above form is generally valid.

Now, explicitly allowing for the presence of an external field $V(\mathbf{x})$, we write $\Omega = -\Gamma - V(\mathbf{x})$, so that our description is governed by the two equations

$$\frac{(\nabla\varphi)^2}{2m} + V + \Gamma = -\varphi_{,t}, \qquad \nabla \cdot (r^2 \nabla\varphi) + m r^2_{,t} = 0 \tag{21}$$

where Γ is an invariant dependent on the system state.

The first equation above will be referred to as the dynamic equation. Having in mind a proof in stages, we *momentarily* restrict Γ to the form

$$\Gamma = F(r, \varphi) + \mu\, \nabla\varphi \cdot \nabla r + \nu\, \nabla\varphi \cdot \nabla\varphi + \eta\, \nabla r \cdot \nabla r + \sigma\, \nabla^2 r + \lambda\, \nabla^2\varphi \tag{22}$$

where the coefficients μ, ν, η, σ, and λ are functions of only r and φ. We further require that the scalar function $F(r, \varphi)$ is defined in the neighborhood of $r = 0$, since otherwise the above equation becomes singular there. The above expression for Γ is the most general invariant form possible involving derivatives through second order and degree. Later, the restriction on Γ as to order and degree will be dropped.

Continuing, we first will show that the requirement that these equations satisfy the superposition principle and also have a bounded probability density drastically limits Γ. In fact, we will show that, of the various coefficients in Γ, only σ does not thereby vanish, and its form is uniquely determined, so that the pair of equations (21) become equivalent to the Schrödinger equation. Second, we shall show that all this remains true even if the restriction on Γ is dropped.

Before beginning, we stress an important point about the form of the first of equations (21) [and thereby also about equation (17)]. In writing the equation this way we mean that the term $(\nabla\varphi)^2/2m$ should survive; i.e., no term in Γ can cancel it out. This means that the coefficient ν in Γ cannot equal $-1/2m$. This must be so, since otherwise equation (17) is replaced by or reduced to the form $\partial_t \varphi = \Omega$; and $\mathbf{v}$ itself has then ceased to be a fundamental quantity.

3.1. Equations of Superposition

We now require that equations (21) satisfy the superposition principle. For this we use exactly the same approach that we used with the classical stochastic theory discussed in Section 2, namely we consider two solutions (r_1, φ_1) and (r_2, φ_2) and we require that there exists another solution (r, φ), where $r = r(r_1, r_2, \varphi_1, \varphi_2)$ and $\varphi = \varphi(r_1, r_2, \varphi_1, \varphi_2)$, and where r and φ are universal functions of their arguments. We are to recall that each solution

(r_1, φ_1), (r_2, φ_2) satisfies equations of exactly the same form as (21) and with the same coefficients μ, ν, η, σ, and λ, but of the arguments (r_1, φ_1) or (r_2, φ_2), respectively. Again, we have equations (4) and (5) relating the solutions. Then inserting the expressions from (4) and (5) into (21) and then, as in Section 2, reexpressing the $\varphi_{i,t}$ and $r_{i,t}$ terms on the right-hand side here back in terms of spatial gradients, we obtain the following relations coming from the dynamic and continuity equations, respectively (using the summation convention)

$$\nabla\varphi_i \cdot \nabla\varphi_j \mathscr{A}_{ij} + \nabla r_i \cdot \nabla r_j \mathscr{B}_{ij} + \nabla\varphi_i \cdot \nabla r_j \mathscr{C}_{ij} + 2mV(1-\alpha_1-\alpha_2)$$
$$+\nabla^2\varphi_i \mathscr{D}_i + \nabla^2 r_i \mathscr{E}_i + F(r, \varphi) - \sum_1^2 \alpha_i F(r_i, \varphi_i) = 0 \tag{23}$$

where

$$\mathscr{A}_{ij} \equiv \alpha_i\alpha_j + 2mA_{ij} + 2m\sigma\frac{\partial A_i}{\partial\varphi_j} + 2m\lambda\frac{\partial\alpha_i}{\partial\varphi_j} - \alpha_i\delta_{ij} - 2m\alpha_i\nu_i\delta_{ij} \tag{24}$$

$$A_{ij} \equiv \mu\alpha_i A_j + \nu\alpha_i\alpha_j + \eta A_i A_j \tag{25}$$

$$\mathscr{B}_{ij} \equiv \beta_i\beta_j + 2mc_{ij} + 2m\sigma\frac{\partial B_i}{\partial r_j} + 2m\lambda\frac{\partial\beta_i}{\partial r_j} - 2m\alpha_i\eta_i\delta_{ij} \tag{26}$$

$$c_{ij} \equiv \mu\beta_i B_j + \nu\beta_i\beta_j + \eta B_i B_j \tag{27}$$

$$\mathscr{C}_{ij} \equiv 2\alpha_i\beta_j + 2mB_{ij} + 2m\sigma\frac{\partial A_i}{\partial r_j} + 2m\lambda\frac{\partial\alpha_i}{\partial r_j} + 2m\sigma\frac{\partial B_j}{\partial\varphi_i}$$
$$+2m\lambda\frac{\partial\beta_j}{\partial\varphi_i} - 2m\alpha_i\mu_i\delta_{ij} - 2\beta_i\delta_{ij} \tag{28}$$

$$B_{ij} \equiv \mu\alpha_i B_j + \mu\beta_j A_i + 2\alpha_i\beta_j\nu + 2\eta A_i B_j \tag{29}$$

$$\mathscr{D}_i \equiv 2m(\sigma A_i + \lambda\alpha_i) - 2m\alpha_i\lambda_i - \beta_i r_i \tag{30}$$

$$\mathscr{E}_i \equiv 2m(\sigma B_i + \lambda\beta_i - \alpha_i\sigma_i) \tag{31}$$

and where

$$\nu_i \equiv \nu(r_i, \varphi_i), \qquad \eta_i \equiv \eta(r_i, \varphi_i), \ldots \tag{32}$$

Further,

$$\nabla\varphi_i \cdot \nabla\varphi_j a_{ij} + \nabla r_i \cdot \nabla r_j b_{ij} + \nabla\varphi_i \cdot \nabla r_j e_{ij} - V(A_1 + A_2)$$
$$+\nabla^2\varphi_i d_i + \nabla^2 r_i f_i = 0 \tag{33}$$

where

$$a_{ij} \equiv \frac{r}{2m}\frac{\partial \alpha_i}{\partial \varphi_j} + \frac{1}{m}\alpha_i A_j - \frac{1}{2m}A_i\delta_{ij} - A_i\nu_i\delta_{ij} \tag{34}$$

$$b_{ij} \equiv \frac{r}{2m}\frac{\partial \beta_i}{\partial r_j} + \frac{1}{m}\beta_i B_j - A_i\eta_i\delta_{ij} \tag{35}$$

$$e_{ij} \equiv \frac{r}{2m}\frac{\partial \alpha_i}{\partial r_j} + \frac{r}{2m}\frac{\partial \beta_j}{\partial \varphi_i} + \frac{1}{m}\alpha_i B_j + \frac{1}{m}\beta_j A_i - A_i\mu_i\delta_{ij} - \frac{1}{m}B_i\delta_{ij} \tag{36}$$

$$d_i \equiv \frac{r}{2m}\alpha_i - A_i\lambda_i - \frac{1}{2m}B_i r_i \tag{37}$$

$$f_i \equiv \frac{r}{2m}\beta_i - A_i\sigma_i \tag{38}$$

We now consider consequences of, first, the dynamic implication, equation (23), and then the continuity implication, equation (33), using the same technique for each. Turning to equation (23), we first choose all the r_i, φ_i to be constants (at a given t) as well as $V \equiv 0$, which gives us a universal relation between F and the F_i. Then, allowing $V \not\equiv 0$ and again choosing all the r_i, φ_i as constants implies that the coefficient of the term containing $2mV$ is zero. Again choosing r_i, φ_1 to be constants and $\nabla\varphi_2$ to be a constant vector (all at a given t) implies that the coefficient of the term containing $\nabla\varphi_2 \cdot \nabla\varphi_2$ is zero. Continuing on in this way, since the coefficients of the various gradient terms are universal functions, we see that we can ultimately set each coefficient (or its symmetric sum) to zero, yielding these universal equations:

$$\mathscr{A}_{ii} = 0, \qquad i = 1, 2 \tag{39}$$

$$\mathscr{A}_{ij} + \mathscr{A}_{ji} = 0, \qquad i \neq j \tag{40}$$

$$\mathscr{B}_{ii} = 0, \qquad i = 1, 2 \tag{41}$$

$$\mathscr{B}_{ij} + \mathscr{B}_{ji} = 0, \qquad i \neq j \tag{42}$$

$$\mathscr{C}_{ij} = 0 \qquad \text{all } i, j \tag{43}$$

$$\alpha_1 + \alpha_2 = 1 \tag{44}$$

$$\mathscr{D}_i = 0, \qquad i = 1, 2 \tag{45}$$

$$\mathscr{E}_i = 0, \qquad i = 1, 2 \tag{46}$$

$$F = \sum_1^2 \alpha_i F_i \tag{46'}$$

Since it can be shown that equation (43) is a consequence of all the others, we have 11 differential equations plus one algebraic equation here to content with.[4]

Again, using exactly the same considerations on equation (33) we obtain the additional equations

$$a_{ii} = 0, \qquad i = 1, 2 \tag{47}$$

$$a_{ij} + a_{ji} = 0, \qquad i \neq j \tag{48}$$

$$b_{ii} = 0, \qquad i = 1, 2 \tag{49}$$

$$b_{ij} + b_{ji} = 0, \qquad i \neq j \tag{50}$$

$$e_{ij} = 0, \qquad \text{all } i, j \tag{51}$$

$$A_1 + A_2 = 0 \tag{52}$$

$$d_i = 0, \qquad i = 1, 2 \tag{53}$$

$$f_i = 0, \qquad i = 1, 2 \tag{54}$$

And here, since it can again be shown that equation (51) is a consequence of the other equations, we have 11 additional equations, for a total of 22 differential and one algebraic equation to be considered as characterizing the superposition principle. These 23 equations comprise the *superposition equations*, and the rest of this section will be devoted to their consequences.

3.2. Consequences of the Equations

In a fairly systematic manner we now consider the consequences of the above superposition equations.

To begin with, we can say something of the dependence of r and φ on φ_1 and φ_2, as follows. From equation (52) we have that

$$\frac{\partial r}{\partial \varphi_1} + \frac{\partial r}{\partial \varphi_2} = 0 \tag{55}$$

which implies that

$$r = r(r_1, r_2, \varphi_2 - \varphi_1) \tag{56}$$

Again, from equation (44) we have that

$$\frac{\partial \varphi}{\partial \varphi_1} + \frac{\partial \varphi}{\partial \varphi_2} = 1 \tag{57}$$

[4]The equations $\mathscr{C}_{ij} = 0$ and $e_{ij} = 0$, both for all i, j, can be shown to be a consequence of the other 20 (differential) superposition relations in this sense: Later in this section it will be shown that $\mu = \lambda = \eta = \nu = 0$ as a consequence of the above 20 relations; and at this time it will then be straightforward to show that $\mathscr{C}_{ij} = 0$ and $e_{ij} = 0$ must also be true.

which implies, by an integration, that

$$\varphi = \varphi_1 + f(r_1, r_2, \varphi_1, \varphi_2) \tag{58}$$

for some function f. But we easily see that

$$\frac{\partial f}{\partial \varphi_1} + \frac{\partial f}{\partial \varphi_2} = 0 \tag{59}$$

which implies that $f = f(r_1, r_2, \varphi_2 - \varphi_1)$, so that we have

$$\varphi = \varphi_1 + f(r_1, r_2, \varphi_2 - \varphi_1) \tag{60}$$

This also implies that all the quantities α_i, β_i, A_i, and B_i depend on φ_1 and φ_2 through the combination $\varphi_2 - \varphi_1$. Equations (56) and (60) will be very useful in the following discussion.

Now, we can show that the coefficients μ, η, σ, λ, and ν can depend only on r, as follows. Equation (53) for $i = 1$ implies that

$$\lambda_1 \equiv \lambda(r_1, \varphi_1) = \frac{(1/2m)(r\alpha_1 - B_1 r_1)}{A_1} \tag{61}$$

But as we have just established, all the quantities on the right-hand side here depend on φ_1 and φ_2 as $\varphi_2 - \varphi_1$. Further, the right-hand side cannot depend on r_2, since λ_1 does not. This means that equation (61) has the form

$$\lambda(r_1, \varphi_1) = \mathscr{H}(r_1, \varphi_2 - \varphi_1) \tag{62}$$

for some function $\mathscr{H}$. But this can only be true if $\lambda_1 = \lambda(r_1)$; and likewise, $\lambda_2 = \lambda(r_2)$ and $\lambda = \lambda(r)$ (we recall that all the functions λ are the same). Again, equation (54) implies for $i = 1$ that

$$\sigma_1 \equiv \sigma(r_1, \varphi_1) = r\beta_1/2mA_1 \tag{63}$$

which implies, since the right-hand side here only depends on r_1 and $\varphi_2 - \varphi_1$, that $\sigma_1 = \sigma(r_1)$; and also, $\sigma_2 = \sigma(r_2)$ and $\sigma = \sigma(r)$. Again, equation (47) implies for $i = 1$, and using the same argument, that $\nu_1 = \nu(r_1)$, $\nu_2 = \nu(r_2)$, and $\nu = \nu(r)$. Likewise, equation (49) implies that $\eta_1 = \eta(r_1)$, $\eta_2 = \eta(r_2)$, and $\eta = \eta(r)$. Finally, in the same way, equation (41) implies that $\mu_1 = \mu(r_1)$, $\mu_2 = \mu(r_2)$, and $\mu = \mu(r)$.

Thus, we have $\mu = \mu(r)$, $\eta = \eta(r)$, $\lambda = \lambda(r)$, $\nu = \nu(r)$, and $\sigma = \sigma(r)$.

Now, we shall demonstrate that the coefficient σ in Γ is of special significance, by proving that if $\sigma \equiv 0$, the superposition principle cannot hold. We shall do this by assuming that $\sigma \equiv 0$ and showing that the superposition equations then lead to a contradiction.

Assuming that $\sigma \equiv 0$ (and therefore that $\sigma_1 = 0 = \sigma_2$), we find from equation (54) that $\beta_1 = 0 = \beta_2$, which implies [from equation (60)] that $\varphi = \varphi_1 + f(\varphi_2 - \varphi_1)$, or $\varphi = \varphi_2 + g(\varphi_1 - \varphi_2)$, for some functions f and g. Before proceeding further, we first show that both α_1 and α_2 must be nonzero, as follows. Suppose $\alpha_1 = 0$; then from the form of φ above we conclude that $f = \xi + c$ (where c is a constant and $\xi \equiv \varphi_2 - \varphi_1$), so that $\varphi = \varphi_1 + (\varphi_2 - \varphi_1) + c = \varphi_2 + c$. But by symmetry we must also have that $\varphi = \varphi_1 + c'$ ($c' =$ const), from which we conclude that $\varphi_2 - \varphi_1 =$ const, which is absurd, since φ_1 and φ_2 are independent. Hence, we conclude that $\alpha_1 \alpha_2 \neq 0$. Continuing on, equations (49) imply that (since $\beta_1 = 0 = \beta_2$) $A_1 \eta_1 = 0 = A_2 \eta_2$. So we have $A_1 = 0 = A_2$, or $\eta_1 = 0 = \eta_2$, or both. Assume the former. Then equation (47) implies that

$$\frac{\partial \alpha_1}{\partial \varphi_1} = -\frac{\partial \alpha_2}{\partial \varphi_1} = \frac{\partial \alpha_2}{\partial \varphi_2} = 0$$

which in turn yields $\varphi = \varphi_1 + C(\varphi_2 - \varphi_1) + D$, where $C = C(r_1, r_2)$ and $D = D(r_1, r_2)$. Therefore, $\alpha_1 = 1 - C$ and $\alpha_2 = C$, where $C \neq 0$ and $C \neq 1$. Now, equation (40) yields the relation $2\alpha_1\alpha_2(1 + 2m\nu) = 0$, which, since $\alpha_1 \alpha_2 \neq 0$, implies that $\nu = -1/2m =$ const. But this cannot be, by the proviso just before Section 3.1. Hence, $A_1 = 0 = A_2$ is unacceptable. So, now we consider the possibility $\eta_1 = 0 = \eta_2 = \eta$ (together with $\beta_1 = 0 = \beta_2$ and $\sigma \equiv 0$, of course). Now, equation (48) gives the relation

$$\frac{r}{m} \frac{\partial \alpha_1}{\partial \varphi_2} = -\frac{1}{m} (\alpha_1 A_2 + \alpha_2 A_1)$$

and equation (40) yields

$$2\alpha_1\alpha_2 + 2m(\mu\alpha_1 A_2 + \mu\alpha_2 A_1 + 2\nu\alpha_1\alpha_2) + 4m\lambda \frac{\partial \alpha_1}{\partial \varphi_2} = 0$$

Combining these two relations then gives

$$\alpha_1 \alpha_2 = \frac{\mu m r - 2m\lambda}{1 + 2m\nu} \frac{\partial \alpha_1}{\partial \varphi_2} \tag{64}$$

where, necessarily, $1 + 2m\nu \neq 0$. Now, $\partial\alpha_1/\partial\varphi_2 = 0$ is impossible, since this would lead to $\alpha_1\alpha_2 = 0$, which has been shown to be untrue. Hence, we conclude that

$$\frac{\alpha_1 \alpha_2}{\partial\alpha_1/\partial\varphi_2} = \frac{\mu m r - 2m\lambda}{1 + 2m\nu} \tag{65}$$

On the left-hand side here we have a function of $(\varphi_2 - \varphi_1)$ only, and on the right-hand side a function of r only. This implies that $r = r(\varphi_2 - \varphi_1)$. But this cannot be, since r (when $r_2 = 0$) $= r_1$ contradicts this. Hence, $\eta_1 = 0 = \eta_2 = \eta$ is also not possible, and we conclude that if superposition holds, then we must have $\sigma \not\equiv 0$. This also implies that in future discussion division by σ is admissible.

We will now go through a sequence of developments which will lead to the fact that $\nu=0$, as well as to the functional form of α_i, β_i, r^2, and A_i, B_i. Beginning with equation (48) we have

$$r\frac{\partial\alpha_1}{\partial\xi}+(\alpha_1 A_2+\alpha_2 A_1)=0 \tag{66}$$

where $\xi\equiv\varphi_2-\varphi_1$. Recalling the definition of the A_i, we can rewrite this as

$$\frac{\partial\alpha_1}{\partial\xi}+(2\alpha_1-1)\frac{\partial\ln r}{\partial\xi}=0 \tag{67}$$

from which we conclude that the functional form of the α_i is given by

$$\begin{aligned}\alpha_1(r_1, r_2, \xi)&=\frac{1}{r^2}h(r_1, r_2)+\frac{1}{2}\\ \alpha_2(r_1, r_2, \xi)&=-\frac{1}{r^2}h(r_1, r_2)+\frac{1}{2}\end{aligned} \tag{68}$$

for some function $h(r_1, r_2)$. Further, $h(r_1, r_2)=-h(r_2, r_1)$.[5] Next, equation (47) implies the relation

$$\frac{\partial\alpha_1}{\partial\xi}+(2\alpha_1-1)\frac{\partial\ln r}{\partial\xi}-\frac{2m}{r}A_1\nu_1=0 \tag{69}$$

which, upon comparison with equation (67), implies that $A_1\nu_1=0=A_2\nu_2$. We have already seen that we cannot have $A_1=0=A_2$, so we conclude that

$$\nu_1=\nu_2=\nu=0 \tag{70}$$

Next, we obtain the functional form of some other relevant quantities. From equation (49) we have

$$\frac{\partial\ln\beta_1}{\partial r_1}+\frac{\partial\ln r^2}{\partial r_1}-\frac{2m}{r}\frac{A_1}{\beta_1}\eta_1=0 \tag{71}$$

Equation (54) implies that $(2m/r)A_1/\beta_1=1/\sigma_1$, which, when combined with the above equation, yields

$$\frac{\partial}{\partial r_1}\ln(\beta_1 r^2)=\frac{\eta_1}{\sigma_1} \tag{72}$$

[5]We have, by interchanging 1 and 2, that $\varphi=\varphi_1+f(r_1, r_2, \xi)=\varphi_2+f(r_2, r_1, -\xi)$. This implies that $f(r_2, r_1, -\xi)=f(r_1, r_2, \xi)-\xi$. We have then that $\alpha_1=1-\partial f/\partial\xi=h/r^2+\frac{1}{2}$, implying that $h/r^2=\frac{1}{2}-(\partial/\partial\xi)f(r_1, r_2, \xi)$. Interchanging 1 and 2 in this equation gives

$$\frac{h(r_2, r_1)}{r^2}=\frac{1}{2}-\frac{\partial}{\partial(-\xi)}f(r_2, r_1, -\xi)=-\frac{1}{2}+\frac{\partial f(r_1, r_2, \xi)}{\partial\xi}=-\frac{h(r_1, r_2)}{r^2}$$

by the above relation. Thus, we have that $h(r_2, r_1)=-h(r_1, r_2)$.

which finally gives the relation

$$\beta_1 = H(r_1)g(r_2, \xi)/r^2 \tag{73}$$

where $\ln H(r_i) \equiv \int (\eta_i/\sigma_i)\, dr_i$ [$\eta_i \equiv 0$ implies that $H(r_i) = \text{const}$] and where $g(r_2, \xi)$ is some function of r_2 and $\varphi_2 - \varphi_1$. Of course, we similarly have the relation

$$\beta_2 = H(r_2)k(r_1, \xi)/r^2 \tag{74}$$

where k is some as yet unspecified function. We can use these results to get the functional form of r^2 as follows. Again using equation (54), we can reexpress equation (73) as

$$\frac{H(r_1)g(r_2, \xi)}{2m} = -\frac{\sigma_1}{2}\frac{\partial r^2}{\partial \xi} \tag{75}$$

from which we conclude that

$$r^2 = \frac{H(r_1)}{m\sigma(r_1)} G(r_2, \xi) + \tilde{f}(r_1, r_2) \tag{76}$$

where $G(r_2, \xi) \equiv -\int g(r_2, \xi)\, d\xi$, and $\tilde{f}$ is some as yet unspecified function. In a similar way we also find that

$$r^2 = \frac{H(r_2)}{m\sigma(r_2)} K(r_1, \xi) + \tilde{g}(r_1, r_2) \tag{77}$$

where $K(r_1, \xi) \equiv \int k(r_1, \xi)\, d\xi$, and $\tilde{g}$ is another unspecified function. Further, from equation (54) we see that $\beta_1/\sigma_1 = -\beta_2/\sigma_2$, which implies that $K = G(\sigma_2/\sigma_1)H_1/H_2$ [where $H_i = H(r_i)$]. Therefore, equation (77) can be rewritten as

$$r^2 = \frac{H(r_1)G}{m\sigma(r_1)} + \tilde{g} \tag{78}$$

which, when compared with equation (76), tells us that $\tilde{g} = \tilde{f}$. We can draw another conclusion from the above relation between K and G. We note that

$$\frac{\sigma(r_2)}{H(r_2)} G(r_2, \xi) = \frac{\sigma(r_1)}{H(r_1)} K(r_1, \xi) \tag{79}$$

This relation can only be true if each side is solely a function of ξ, since otherwise it would imply that $r_1 = r_1(r_2, \xi)$, which is untrue, since r_1, r_2, and ξ are independent. So, we can write

$$\frac{\sigma(r_1)}{H(r_1)} K(r_1, \xi) = N(\xi) \tag{80}$$

for some function $N(\xi)$. Therefore, we can write the general functional form of r^2 as

$$r^2 = \frac{H(r_1)H(r_2)}{m\sigma(r_1)\sigma(r_2)} N(\xi) + \tilde{f}(r_1, r_2) \tag{81}$$

where, moreover, we now see that $\tilde{f}(r_1, r_2) = \tilde{f}(r_2, r_1)$.

We can now turn around and get an even more useful expression for β_i. By differentiating the relation between K and G, we obtain

$$-\frac{\sigma_2}{H_2} g = \frac{\sigma_1}{H_1} k = \frac{dN}{d\xi} \equiv N'(\xi)$$

Inserting this into equation (73), we then obtain

$$\beta_1 = -\frac{H(r_1)H(r_2)}{\sigma(r_2)r^2} N'(\xi), \qquad \beta_2 = \frac{H(r_1)H(r_2)}{\sigma(r_1)r^2} N'(\xi) \tag{82}$$

We also find for $A_i = \partial r/\partial \varphi_i = \pm \partial r/\partial \xi$ the expressions

$$A_1 = -\frac{H(r_1)H(r_2)}{2mr\sigma(r_1)\sigma(r_2)} N'(\xi), \qquad A_2 = \frac{H(r_1)H(r_2)}{2mr\sigma(r_1)\sigma(r_2)} N'(\xi) \tag{83}$$

The expressions for $B_i = \partial r/\partial r_i$ are also easy to calculate. We do not bother to express them here, however.

Thus, we have shown that $\nu = 0$, and have obtained the general functional forms of α_i, β_i, r^2, A_i, and B_i.

We can now use the above expressions for β_i to obtain a differential relation between σ and η, as follows. From equation (82) we find, by differentiating with respect to r_2, that

$$\frac{r}{2m}\frac{\partial \beta_1}{\partial r_2} = -\frac{1}{m}\beta_1\beta_2 - \frac{r}{2m}\frac{\beta_1}{\sigma_2}\sigma_2' + \frac{r}{2m}\beta_1\frac{\eta_2}{\sigma_2} \tag{84}$$

where primes denote differentiation. We also have that

$$\frac{r}{2m}\frac{\partial \beta_2}{\partial r_1} = -\frac{\beta_2}{m}B_1 - \frac{r}{2m}\frac{\beta_2}{\sigma_1}\sigma_1' + \frac{r}{2m}\frac{\beta_2}{\sigma_1}\eta_1 \tag{85}$$

Adding these equations gives

$$\frac{r}{2m}\left(\frac{\partial \beta_1}{\partial r_2} + \frac{\partial \beta_2}{\partial r_1}\right) + \frac{1}{m}(\beta_1 B_2 + \beta_2 B_1)$$

$$= \frac{r}{2m}\left(\frac{-\beta_1}{\sigma_2}\sigma_2^1 + \beta_1\frac{\eta_2}{\sigma_2} - \frac{\beta_2}{\sigma_1}\sigma_1' + \frac{\beta_2}{\sigma_1}\eta_1\right) \tag{86}$$

Comparing this with (50), we conclude that

$$\left(-\frac{\beta_1}{\sigma_2}\sigma_2'+\beta_1\frac{\eta_2}{\sigma_2}-\frac{\beta_2}{\sigma_1}\sigma_1'+\frac{\beta_2}{\sigma_1}\eta_1\right)=0 \tag{87}$$

Now, using the familiar relation $\beta_1/\sigma_1=-\beta_2/\sigma_2$ here gives the result

$$\frac{1}{\sigma_2}\left(\frac{\eta_2}{\sigma_2}-\frac{\sigma_2'}{\sigma_2}\right)=\frac{1}{\sigma_1}\left(\frac{\eta_1}{\sigma_1}-\frac{\sigma_1'}{\sigma_1}\right) \tag{88}$$

from which we conclude that, necessarily,

$$\frac{1}{\sigma}\left(\frac{\eta}{\sigma}-\frac{\sigma'}{\sigma}\right)=\text{const}\equiv c \tag{89}$$

And from this we get the equation

$$\sigma'+c\sigma^2-\eta=0 \tag{90}$$

which is a Riccati equation relating σ and η. Of special importance here is the implication that

$$H=\sigma\exp\left(c\int\sigma\,dr\right) \tag{91}$$

Next, we proceed to show that $\mu=0$. Noting that

$$\frac{\partial A_1}{\partial\varphi_1}=-\frac{\partial A_2}{\partial\varphi_1}=\frac{\partial A_2}{\partial\xi}=\frac{\partial A_2}{\partial\varphi_2};\qquad \frac{\partial\alpha_1}{\partial\varphi_1}=-\frac{\partial\alpha_2}{\partial\varphi_1}=\frac{\partial\alpha_2}{\partial\xi}=\frac{\partial\alpha_2}{\partial\varphi_2} \tag{92}$$

we find by subtracting the two equations (39) from each other that

$$\alpha_1^2-\alpha_2^2+2m(A_{11}-A_{22})=\alpha_1-\alpha_2 \tag{93}$$

since $\nu=0$. Now, $A_{11}-A_{22}=-\mu A_2$, which, when inserted into the above equation, yields $\mu A_2=0$. Since A_2 cannot be zero, as already shown, we conclude that

$$\mu=\mu_1=\mu_2=0 \tag{94}$$

We now make some considerations that will allow us to conclude that $\lambda=0=\eta$; determine the exact form for σ, and ultimately determine the expression for $N(\xi)$. We begin with (53). We insert into this equation the expression for α_1 given in (68), the expression for A_1 given in (83), and the expression for $B_1=\partial r/\partial r_1$ found by differentiating (81). Then, multiplying through by r^2 and collecting terms gives the relation

$$\frac{N(\xi)}{2m}\left(\frac{H_1H_2}{\sigma_1\sigma_2}-\frac{r_1H_1H_2\eta_1}{\sigma_1^2\sigma_2}+\frac{r_1H_1H_2\sigma_1'}{\sigma_1^2\sigma_2}\right)+\frac{\lambda_1H_1H_2}{\sigma_1\sigma_2}N'(\xi)$$
$$+\left(\frac{1}{2}\tilde{f}+h-\frac{r_1}{2}\frac{\partial\tilde{f}}{\partial r_1}\right)=0 \tag{95}$$

for all r_1, r_2, and ξ. This equation has the form

$$N(\xi)M(r_1, r_2) + N'(\xi)P(r_1, r_2) + Q(r_1, r_2) = 0$$

for all r_1, r_2, and ξ. Differentiating with respect to ξ gives the relation $N'M + N''P = 0$ for all r_1, r_2, and ξ. Now, if $P \not\equiv 0$, we may divide through by P to conclude that $N''/N' = \text{const} \equiv \Lambda = -M/P$. This yields the solution for $N(\xi)$ as

$$N(\xi) = C\,e^{\Lambda\xi} + D \tag{96}$$

where C and D are constants. We note that $C = 0$ is unacceptable here, since it implies that $N' = 0$, which implies that $A_1 = 0 = A_2$. However, we see that, regardless of the sign of Λ, $N \to \infty$ as $\Lambda\xi \to \infty$. This is inadmissible, since it implies that for $\Lambda(\varphi_2 - \varphi_1) \to \infty$ (say, for certain choices of ξ, when $|\mathbf{x}| \to \infty$) the function r^2 increases without limit; i.e., the probability density is unbounded. So, this solution must be rejected, and we conclude that we must have that $P \equiv 0$, i.e., that

$$\lambda_1 = 0 = \lambda_2 = \lambda \tag{97}$$

Continuing, with $P \equiv 0$, we now have to satisfy the simple relation $NM + Q = 0$. Differentiating this with respect to ξ then implies that $N'M = 0$, implying that $N'(\xi) \equiv 0$ and/or $M \equiv 0$. Now, $N'(\xi) = 0$ is unacceptable, since it implies that $A_1 = 0 = A_2$. So, we conclude that $M \equiv 0$, which then also implies that $Q \equiv 0$. Now, $M \equiv 0$ implies that $1 - r_1\eta_1/\sigma_1 + r_1\sigma_1'/\sigma_1 = 0$. Combining this relation with (90) gives $1 - cr_1\sigma_1 = 0$; that is,

$$\sigma = 1/cr \tag{98}$$

where $c^{-1} = \text{const} \neq 0$. Furthermore, we now find η from (90) to be given by

$$\eta = 0 = \eta_1 = \eta_2 \tag{99}$$

Collecting our latest results, we have shown that

$$\sigma = 1/cr, \qquad \eta = \lambda = \mu = \nu = 0 \tag{100}$$

as the result of the requirement of superposition, and bounded probability density. Finally, $Q \equiv 0$ gives the relations

$$\frac{1}{2}\tilde{f} + h - \frac{r_1}{2}\frac{\partial \tilde{f}}{\partial r_1} = 0, \qquad \frac{1}{2}\tilde{f} - h - \frac{r_2}{2}\frac{\partial \tilde{f}}{\partial r_2} = 0 \tag{101}$$

the second equation following from the first by $1 \leftrightarrow 2$ interchange, and the already established antisymmetry of $h(r_1, r_2)$. Adding the two equations above then gives an equation for $\tilde{f}$,

$$\tilde{f} = \frac{r_1}{2}\frac{\partial \tilde{f}}{\partial r_1} + \frac{r_2}{2}\frac{\partial \tilde{f}}{\partial r_2} \tag{102}$$

which only informs us that $\tilde{f}$ is homogeneous of degree 2. We do get a useful result by subtracting the above two equations. This gives a relation determining h from $\tilde{f}$ as

$$h = \frac{1}{4}\left(r_1 \frac{\partial \tilde{f}}{\partial r_1} - r_2 \frac{\partial \tilde{f}}{\partial r_2}\right) \tag{103}$$

3.3. Determining $\tilde{f}$ and $N(\xi)$

We shall now determine $\tilde{f}$ and $N(\xi)$. We note in this connection that none of the 22 differential equations of superposition give any further information on $\tilde{f}$ and $N(\xi)$, except for essentially equations (39), (41), and (42).

We begin with (39), which gives us

$$\alpha_1^2 + 2m\sigma\, \partial A_1/\partial \varphi_1 = \alpha_1 \tag{104}$$

from which we obtain

$$\alpha_1 \alpha_2 = -2m\sigma\, \partial A_1/\partial \xi \tag{105}$$

which also follows from (40). From the preceding expressions for α_i and A_j, the above equation can be rewritten as

$$\frac{1}{4} - \frac{h^2}{r^4} = \frac{c r_1 r_2 N''(\xi)}{r^2} \tag{106}$$

and using (81) for r^2 here, we finally obtain the relation

$$h^2(r_1, r_2) = \frac{1}{4}\left(\frac{N^2}{m^2\sigma_1^2\sigma_2^2} + \tilde{f}^2 + \frac{2\tilde{f}N}{m\sigma_1\sigma_2}\right) - \frac{c r_1 r_2 N N''}{m\sigma_1\sigma_2} - c r_1 r_2 \tilde{f} N'' \tag{107}$$

where primes signify derivatives, and the relation is true for all r_1, r_2, and ξ. This equation has the form

$$\begin{aligned} &A(r_1, r_2)\gamma(\xi) + B(r_1, r_2)\delta(\xi) + C(r_1, r_2)\varepsilon(\xi) \\ &\quad + D(r_1, r_2)\theta(\xi) + E(r_1, r_2) = h^2(r_1, r_2) \end{aligned} \tag{108}$$

for all r_1, r_2, ξ; and where $\gamma \equiv N^2$, $\delta \equiv N$, $\varepsilon \equiv NN^n$, and $\theta \equiv N^n$. We now seek the implications of this relation. Taking derivatives, we have

$$A\gamma' + B\delta' + C\varepsilon' + D\theta' = 0 \tag{109}$$

for all r_1, r_2, ξ. Now, $\gamma' \not\equiv 0$ (since $N \equiv 0$ or $N' \equiv 0$ implies that $A_1 = 0 = A_2$), so we have

$$A + B\frac{\delta'}{\gamma'} + C\frac{\varepsilon'}{\gamma'} + D\frac{\theta'}{\gamma'} = 0 \tag{110}$$

Another differentiation gives us

$$B\left(\frac{\delta'}{\gamma'}\right)' + C\left(\frac{\varepsilon'}{\gamma'}\right)' + D\left(\frac{\theta'}{\gamma'}\right)' = 0 \tag{111}$$

for all r_1, r_2, ξ. Now, we can easily show that $(\delta'/\gamma')' \not\equiv 0$, since $N' \not\equiv 0$ and $N \not\equiv 0$. So we can divide the above equation through by this factor. Again we take derivatives and are then confronted with the equation

$$C\left[\frac{(\varepsilon'/\gamma')'}{(\delta'/\gamma')'}\right]' + D\left[\frac{\theta'/\gamma')'}{(\delta'/\gamma')'}\right]' = 0 \tag{112}$$

If the first bracket term here is zero, we get the consequence that

$$NN'' = \Lambda_1 N + \Lambda_2 N^2 + \Lambda_3$$

where the Λ_i are constants, and if the second bracket term is zero, we get the result that

$$NN'' = M_1 N^2 + M_2 N^3 + M_3 N$$

where the M_i are constants. We cannot have both relations holding, since that implies that $N = \text{const}$, which implies that $N' = 0$, implying that $A_1 = 0 = A_2$. So, we must consider separately the two cases coming from one or the other bracket being nonzero. Now, if the first bracket term is assumed to be nonzero, we are forced to the conclusion (after an additional differentiation) that either $D \equiv 0$ or the derivative of the quotient of the brackets is zero. Now, $D \equiv 0$ implies that $\tilde{f} \equiv 0$ (since $c \neq 0$), implying that $h \equiv 0$, which yields $\alpha_1 = \frac{1}{2} = \alpha_2$, which is unacceptable, since it implies that $N \to \infty$ with ξ, so that the probability density is unbounded.[6] We are forced then to the alternative choice, which can then be shown to imply the relation

$$N'' = \Lambda_1 NN'' + \Lambda_2 N + \Lambda_3 N^2 + \Lambda_4 \tag{113}$$

[6]Now, $\varphi = \varphi_1 + f(r_1, r_2, \xi)$ and $\alpha_1 = 1 - \partial f/\partial\xi$. Therefore, if $\alpha_1 = \frac{1}{2} = \alpha_2$ (for all r_1, r_2), $\partial f/\partial\xi = \frac{1}{2}$, implying that $f = \frac{1}{2}\xi + l(r_1, r_2)$ for some function l. Then, $\varphi = \frac{1}{2}(\varphi_1 + \varphi_2) + l$, and then $\beta_1 = \partial l/\partial r_1 =$ a function of r_1, r_2. Therefore, $\beta_1 = -(cr_2 H_1 H_2/r^2)N'(\xi)$ is also a function of r_1, r_2. This implies that $N'(\xi)/r^2 =$ a function of r_1, r_2, so we may write $r^2 = p(r_1, r_2)N'(\xi)$ for some function $p(r_1, r_2)$. Now we have $A_1 = \partial r/\partial\varphi_1 = -(1/2r)p(r_1, r_2)N''(\xi)$. Further, equation (22) gives $\sigma A_1 = r_1\beta_1/2m$, implying that $A_1 = -(c^2 r_1 r_2 H_1 H_2/2mr)N(\xi)$. Comparing this with the above expression for A_1, we conclude that

$$\frac{c^2 r_1 r_2 H_1 H_2}{r} N'(\xi) = \frac{1}{2r} pN''$$

implying that $N'' = \Lambda N'$ for some constant Λ, where $\Lambda \neq 0$, since $p \to \infty$ or $c \to 0$ is not permissible, from earlier discussions. So we have $N(\xi) = ae^{\Lambda\xi} + b$ for some constants a and b, where a cannot be zero, since that would imply that $N'(\xi) = 0$, which implies that $A_1 = 0 = A_2$. Thus, we have that $N(\xi) \to \infty$ as $\Lambda\xi \to \infty$, giving an unbounded probability density, which we consider unacceptable.

where the Λ_i are constants. If, instead, the second bracket term in (112) is assumed to be nonzero, we face the choice of either $C \equiv 0$ or the derivative of the inverse ratio of the brackets being zero. $C \equiv 0$ is ruled out, since it implies that $c = 0$, and the alternative then yields the relation

$$NN'' = S_1 N'' + S_2 N + S_3 N^2 + S_4$$

where the S_i are constants; and we see that this is really the same as (113), resulting from the other alternative. Thus we have so far a *single* possibility, which we express as $\theta = \Lambda_1 \varepsilon + \Lambda_2 \delta + \Lambda_3 \gamma + \Lambda_4$. Now, inserting the derivative of this expression into (109) gives the relation

$$(A + D\Lambda_3) + (B + D\Lambda_2)\frac{\delta'}{\gamma'} + (C + D\Lambda_1)\frac{\varepsilon'}{\gamma'} = 0 \tag{114}$$

since $\gamma' \neq 0$. Repeating the previous process, we take derivatives here, giving

$$(B + D\Lambda_2)\left(\frac{\delta'}{\gamma'}\right)' + (C + D\Lambda_1)\left(\frac{\varepsilon'}{\gamma'}\right)' = 0 \tag{115}$$

We have already established that $\delta'/\gamma')' \neq 0$, so we obtain the relation, after one more differentiation,

$$(C + D\Lambda_1)\left[\frac{(\varepsilon'/\gamma')'}{(\delta'/\gamma')'}\right]' \equiv 0 \tag{116}$$

Once again we are faced with the alternatives that either $(C + D\Lambda_1) \equiv 0$ and/or the bracket term is zero. The first choice can be shown to imply that $\tilde{f} = -c^2 r_1 r_2/\Lambda_1 m$. But this implies that [see (103)] $h = 0$, which implies that $\alpha_1 = \frac{1}{2} = \alpha_2$, which has been shown in footnote 6 to be unacceptable. So, we must have the bracket term vanishing, and this yields the relation $\varepsilon' = \tilde{\Lambda}_1 \delta' + \tilde{\Lambda}_2 \gamma'$ for constants $\tilde{\Lambda}_1$, $\tilde{\Lambda}_2$. Putting this back into (114) then gives, after one more differentiation, the relation

$$(B + D\Lambda_2 + \tilde{\Lambda}_1 C + \tilde{\Lambda}_1 \Lambda_1 D)(\delta'/\gamma')' \equiv 0 \tag{117}$$

Now here we only have one choice, since the bracket term has already been shown to be nonzero. Then, equating the expression in parentheses to zero gives the relation

$$\tilde{f} = \frac{\tilde{\Lambda}_1 c^2 r_1 r_2}{m(c/2m - \Lambda_2 - \Lambda_1 \tilde{\Lambda}_1)} \tag{118}$$

But this expression for $\tilde{f}$ seems unacceptable, since it apparently implies that $h = 0$, which implies that $\alpha_1 = \frac{1}{2} = \alpha_2$. On the other hand, this expression must be correct, since it is our *only* choice. Thus, it must be that both numerator and denominator in the above expression are zero. So, we conclude that $\tilde{\Lambda}_1 = 0$ and $\Lambda_2 = c/2m$. And we then can write, after an integration,

$$\varepsilon = \tilde{\Lambda}_2 \gamma + \tilde{\Lambda}_3 \tag{119}$$

where $\tilde{\Lambda}_2$, $\tilde{\Lambda}_3$ are constants. In terms of N we then have the relation

$$NN'' = \tilde{\Lambda}_2 N^2 + \tilde{\Lambda}_3 \tag{120}$$

as a necessary consequence of equation (104). This, then, is the equation governing $N(\xi)$. We note that if $\tilde{\Lambda}_3 \neq 0$, we have a difficult nonlinear differential equation to solve. However, we are saved from this problem by the following consistency consideration. Returning to (108) [from which (120) followed], we rewrite it by substituting in for θ the expression given by (120), which then yields

$$AN^2 + BN + C(\tilde{\Lambda}_1 N^2 + \tilde{\Lambda}_2) + DN'' = h^2 - E \tag{121}$$

From this we obtain, by another differentiation followed by division by $N' \neq 0$, that

$$2N(A + C\tilde{\Lambda}_1) + B + \frac{N'''}{N'} D = 0 \tag{122}$$

One more differentiation and division by N' then yields the relation

$$\frac{2(A + C\tilde{\Lambda}_1)}{D} = -\frac{1}{N'}\left(\frac{N'''}{N'}\right)' \tag{123}$$

since $D \neq 0$. From this relation we conclude that both sides must be constant, which then implies the relation $N''' = MN'N + SN'$, where M and S are constants. Finally, integrating this expression and multiplying it by N gives

$$NN'' = \tilde{M}N^3 + SN^2 + TN \tag{124}$$

where $\tilde{M}$, S, and T are constants. Now, this equation must be consistent with (120), since it is a consequence of (120) inserted into (108). But this consistency can only be achieved, as is seen by comparison, if $\tilde{M} \equiv 0$, $T \equiv 0$, and $\tilde{\Lambda}_3 \equiv 0$. That is, we must, in fact, have the equation on $N(\xi)$ as

$$NN'' = \alpha N^2 \tag{125}$$

for some constant α. And dividing through by N (which must be nonzero, as we have seen before) finally gives, as the equation determining $N(\xi)$,

$$N'' = \alpha N \tag{126}$$

where it must be that the constant $\alpha < 0$, otherwise we would have an exponential solution, thereby producing an unbounded probability density. Thus, we can write

$$N(\xi) = a \cos(\sqrt{-\alpha}\,\xi + b) \tag{127}$$

for some constants a and b, as the only possible solution for $N(\xi)$ compatible with our requirements. We now have

$$r^2 = c^2 A r_1 r_2 \cos(\sqrt{-\alpha}\,\xi + b) + \tilde{f}(r_1, r_2) \tag{128}$$

where $A \equiv H_1 H_2 a/m$.

To complete our determination of $\tilde{f}$ and $N(\xi)$, we employ one of the very few equations that so far has not been used, namely the relation

$$\beta_1^2 + 2m\sigma\,\partial B_1/\partial r_1 = 0 \tag{129}$$

coming from (41). Now,

$$\beta_1 = \frac{\partial \varphi}{\partial r_1} = \frac{Amcr_2}{r^2}\sqrt{-\alpha}\,\sin(\sqrt{-\alpha}\,\xi + b) \tag{130}$$

and

$$B_1 = \frac{c^2 A}{2r} r_2 \cos(\sqrt{-\alpha}\,\xi + b) + \frac{1}{2r}\frac{\partial \tilde{f}}{\partial r_1}. \tag{131}$$

and

$$\frac{\partial B_1}{\partial r_1} = -\frac{1}{4r^3}\left[c^2 A r_2 \cos(\sqrt{-\alpha}\,\xi + b) + \frac{\partial \tilde{f}}{\partial r_1}\right]^2 + \frac{1}{2r}\frac{\partial^2 \tilde{f}}{\partial r_1^2} \tag{132}$$

Putting these expressions into (129) then gives (with $k \equiv \sqrt{-\alpha}\,\xi + b$) the relation

$$M(r_1, r_2) + P(r_1, r_2)\cos^2 k + Q(r_1, r_2)\cos k = 0 \tag{133}$$

for all r_1, r_2, and ξ where

$$M \equiv -\alpha A^2 m^2 c^2 r_2^2 - \frac{m}{2c}\left(\frac{\partial \tilde{f}}{\partial r_1}\right)^2 + \frac{m}{c}\tilde{f}\frac{\partial^2 \tilde{f}}{\partial r_1^2} \tag{134}$$

$$P \equiv \alpha A^2 m^2 c^2 r_2^2 - \frac{m}{2} c^3 A^2 r_2^2 \tag{135}$$

$$Q \equiv -mcAr_2 \frac{\partial \tilde{f}}{\partial r_1} + mcAr_1 r_2 \frac{\partial^2 \tilde{f}}{\partial r_1^2} \tag{136}$$

Now, choosing k so that $\cos k = 0$, gives $M \equiv 0$. Then dividing the remainder through by $\cos k$ (for $\cos k \neq 0$) and then differentiating with respect to ξ obviously implies that also $P \equiv 0$ and $Q \equiv 0$. Considering $Q \equiv 0$ first, we have the relations

$$r_1 \frac{\partial^2 \tilde{f}}{\partial r_1^2} = \frac{\partial \tilde{f}}{\partial r_1}, \qquad r_2 \frac{\partial^2 \tilde{f}}{\partial r_2^2} = \frac{\partial \tilde{f}}{\partial r_2} \tag{137}$$

the second equation here resulting by replacing (129) with $\beta_2^2 + 2m\sigma\, \partial B_2/\partial r_2 = 0$. It is not difficult to show that the general solution to these equations is of the form $\tilde{f} = \Lambda(r_1^2 + r_2^2) + \Lambda_0$, where Λ and Λ_0 are constants.[7] Further, from the requirement that $r^2(r_2 = 0) = r_1^2$ we see that $\Lambda_0 = 0$ and $\Lambda = 1$. Thus we have

$$\tilde{f} = r_1^2 + r_2^2 \tag{138}$$

and $\tilde{f}$ has been determined.

Further, we now have from (103) that

$$h = \tfrac{1}{2}(r_1^2 - r_2^2) \tag{139}$$

and therefore

$$\alpha_1 = \frac{r_1^2 - r_2^2}{2r^2} + \tfrac{1}{2}; \qquad \alpha_2 = -\frac{(r_1^2 - r_2^2)}{2r^2} + \tfrac{1}{2} \tag{140}$$

and also

$$r^2 = c^2 A r_1 r_2 \cos k + (r_1^2 + r_2^2) \tag{141}$$

Next, considering $P \equiv 0$, we obtain the relation

$$\alpha = c/2m \tag{142}$$

Finally, $M \equiv 0$ yields the relation

$$\tilde{f}\frac{\partial^2 \tilde{f}}{\partial r_1^2} - \frac{1}{2}\left(\frac{\partial \tilde{f}}{\partial r_1}\right)^2 = \alpha A^2 c^3 m r_2^2 \tag{143}$$

[7]Briefly, from equation (137) we obtain $\partial^2 \tilde{f}/\partial y^2 = 0$, where $y \equiv r_1^2$. Then we can write $\tilde{f}(r_1, r_2) = r_1^2 g(r_2) + h(r_2)$ for some functions g and h. By symmetry we must also have $\tilde{f}(r_1, r_2) = r_2^2 k(r_1) + l(r_1)$ for some k and l. Equating these two expressions gives $h(0) = l(0)$. By separately placing r_1, r_2 to zero we then conclude that

$$r_1^2 g(r_2) + r_2^2 k(0) + l(0) = r_2^2 k(r_1) + r_1^2 g(0) + h(0)$$

from which we obtain

$$\frac{g(r_2) - g(0)}{r_2^2} = \frac{k(r_1) - k(0)}{r_1^2}$$

for all r_1, r_2. So each side of this equation must be a constant, say α, since r_1, r_2 are independent. If $\alpha \neq 0$, we have $g(r_2) = \alpha r_2^2 + g_0$ and $k(r_1) = \alpha r_1^2 + k_0$. This implies that

$$\tilde{f} = r_2^2(\alpha r_1^2 + k_0) + l(r_1) = \alpha r_1^2 r_2^2 + k_0 r_2^2 + g_0 r_1^2 + h_0$$

But this does not satisfy the condition that $\tilde{f}$ be homogeneous of degree 2 as required by equation (102), unless $\alpha = 0$. So we must have $\alpha = 0$, which then implies, from the above, that $g(r_2) = g(0) \equiv g_0$ and $k(r_1) = k(0) \equiv k_0$. Therefore, $\tilde{f} = r_2^2 k_0 + r_1^2 g_0 + h_0$. Finally, the requirement that $\tilde{f}(r_1, r_2) = \tilde{f}(r_2, r_1)$ [see equation (81)] implies that $k_0 = g_0$ and we have $\tilde{f}(r_1, r_2) = g_0(r_1^2 + r_2^2) + h_0$.

which, together with $\tilde{f} = r_1^2 + r_2^2$, gives the result $\alpha A^2 c^3 m = 2$, which, in turn, together with $\alpha = c/2m$, implies that

$$Ac^2 = +2 \tag{144}$$

where we have adjusted the constant b in k so that the possible negative sign here need not be considered. We now have the relation

$$r^2 = r_1^2 + r_2^2 + 2r_1 r_2 \cos\left[\left(-\frac{c}{2m}\right)^{1/2} \xi + b\right] \tag{145}$$

and $N(\xi)$ has been completely determined.

Now, expressing c as $c = -2m/\hbar^2$ and using (139) to get the α_i, and using the above expression found for $N(\xi)$, which gives the β_i, we have, finally, the following expression for $\varphi = \varphi(r_1, r_2, \xi)$ and $r = r(r_1, r_2, \xi)$:

$$r^2 = r_1^2 + r_2^2 + 2r_1 r_2 \cos\frac{\varphi_2 - \varphi_1}{\hbar} \tag{146}$$

and by an integration

$$\varphi = \sin^{-1}\left\{\frac{r_2 \sin(\varphi_2 - \varphi_1)}{\{r_1^2 + r_2^2 + 2r_1 r_2 \cos[(\varphi_2 - \varphi_1)/\hbar]\}^{1/2}}\right\} + \varphi_1 \tag{147}$$

where we have redefined φ_1 and φ_2 by adding arbitrary constants to each, thereby canceling out the b that would otherwise appear in the argument of the cosine, and also thereby canceling the additive constant that would have appeared after φ_1 above.

We recognize these equations as just the customary superposition relations of conventional quantum mechanics.

3.4. The Hyperbolic Solution

As an essential part of the previous development we used the requirement that $N \to \infty$ with ξ, or $N(\xi) \sim e^{\alpha\xi}$, was not to be allowed, since it leads to $r^2 \to \infty$ for certain limiting ξ; i.e., it leads to an unbounded probability density. Disallowing such solutions forced us, among other things, to the customary superposition relations above of conventional quantum mechanics. It is of interest then to point out that for the choice $N(\xi) = a\cosh(\xi/\hbar)$, $\mu = \nu = \lambda = \eta = 0$, $\sigma = +(\hbar^2/2m)(1/r)$ (i.e., $c = = +2m/\hbar^2$), and $F = \text{const}$ [see (22)], all the 23 equations of superposition are again satisfied and we have superposition holding, with

$$r^2 = r_1^2 + r_2^2 + 2r_1 r_2 \cosh\frac{\varphi_2 - \varphi_1}{\hbar} \tag{148}$$

and

$$\varphi = \sinh^{-1}\left\{\frac{r_2 \sinh[(\varphi_2 - \varphi_1)/\hbar]}{\{r_1^2 + r_2^2 + 2r_1 r_2 \cosh[(\varphi_2 - \varphi_1)/\hbar)]\}^{1/2}}\right\} + \varphi_1 \tag{149}$$

This "alternative" quantum theory is interesting and, as it turns out, displays many of the same qualitative features as does conventional quantum theory. This theory may be said to be characterized by a quantum potential of the opposite sign to that of conventional quantum theory.

Just as conventional quantum theory may be expressed in terms of the variables $\xi \equiv r\cos(\varphi/\hbar)$ and $\eta \equiv r\sin(\varphi/\hbar)$, thereby yielding the pair of real differential equations (equivalent to the Schrödinger equation)

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\xi = -\hbar\,\partial_t\eta; \qquad \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\eta = \hbar\,\partial_t\xi \tag{150}$$

so then can the alternative quantum theory be expressed in terms of the variables $\tilde{\xi} \equiv r\cosh(\varphi/\hbar)$ and $\tilde{\eta} \equiv r\sinh(\varphi/\hbar)$, satisfying the equations

$$\left(\frac{\hbar^2}{2m}\nabla^2 + V\right)\tilde{\eta} = -\hbar\,\partial_t\tilde{\xi}; \qquad \left(\frac{\hbar^2}{2m}\nabla^2 + V\right)\tilde{\xi} = -\hbar\,\partial_t\tilde{\eta} \tag{151}$$

We then see that both descriptions satisfy the linear superposition principle as well. However, conventional quantum theory satisfies the general linear superposition principle, as is easily shown by the substitution $\xi \to \alpha\xi - \beta\eta$, $\eta \to \alpha\eta + \beta\xi$ (for any real α, β), whereas the alternative theory above does not. We note, of course, that in the alternative theory, $r^2 > \infty$ occurs for certain limiting values of ξ, which we regard as inadmissible. However, if one only uses wavefunctions here with bounded φ, and therefore ξ, this problem is averted. We discuss this interesting possibility no further at this point.

3.5. The Function $F(r, \varphi)$

Before finishing our discussion, we must fully dispose of the function Γ. So far we have accounted only for *almost* all of its terms. There remains the matter of the scalar function, $F(r, \varphi)$. With regard to F, then, consider two constituent solutions, where $\varphi_1 = \varphi_2$ and $r_2 = \varepsilon$, where ε is very small. Then (147) implies that $\varphi - \varphi_1 = n\pi$, for some integer n. Redefining φ with an additive constant (which changes no physics) allows us to choose $\varphi = \varphi_1$ here. Further, (146) gives us $r^2 \approx r_1^2(1 + 2\varepsilon/r_1)$ and thus $r \approx r_1(1 + \varepsilon/r_1)$. Now,

$$\alpha_1 = \tfrac{1}{2}[1 + (r_1^2 - r_2^2)/r^2] \approx 1 - \varepsilon/r_1, \qquad \alpha_2 \approx \varepsilon/r_1$$

Looking back we see an additional equation that, so far, has not been stressed. It is equation (46′),

$$F(r, \varphi) = \alpha_1 F(r_1, \varphi_1) + \alpha_2 F(r_2, \varphi_2) \tag{152}$$

For the special case we are now considering, this equation gives

$$F(r_1+\varepsilon, \varphi_1) \approx \left(1-\frac{\varepsilon}{r_1}\right) F(r_1, \varphi_1) + \frac{\varepsilon}{r_1} F(\varepsilon, \varphi_1) \tag{153}$$

Expanding the left-hand side in a Taylor series, we get

$$F(r_1, \varphi_1) + \varepsilon \frac{\partial F}{\partial r_1} \approx \left(1-\frac{\varepsilon}{r_1}\right) F(r_1, \varphi_1) + \frac{\varepsilon}{r_1} F(0, \varphi_1) \tag{154}$$

through first order in ε, where we require that $F(0, \varphi_1)$ is defined as mentioned earlier [see the discussion after equation (22)]. The above equation then gives the relation

$$\frac{\partial F}{\partial r_1} = -\frac{F}{r_1} + \frac{1}{r_1} F(0, \varphi_1) \tag{155}$$

The solution here is easily found to be

$$F(r_1, \varphi_1) = \Lambda(\varphi_1)/r_1 + F(0, \varphi_1) \tag{156}$$

for some function $\Lambda(\varphi_1)$. But this relation implies that, in fact, $F(0, \varphi_1)$ is not defined, which violates the basic requirement just mentioned concerning F. Thus, it must be that $\Lambda(\varphi_1) \equiv 0$, implying that $F = F(\varphi_1)$.

We can now proceed to show that F cannot even depend on φ_1, as follows. Choose $r_1 = r_2$ and $\varphi_2 - \varphi_1 = \varepsilon$. Then the superposition relations already established imply that $\varphi = \varphi_1 + \varepsilon/2$, this being true for *any* ε, small or not [this follows by considering addition of solutions in the form $r\exp(i\sqrt{-\alpha}\varphi)$]. Now, $r_1 = r_2$ implies that $\alpha_1 = \frac{1}{2} = \alpha_2$ (this is acceptable for *particular* r_1, r_2; in contrast to footnote 6), so that (152) now gives, for any φ_1, φ_2

$$F(\varphi_1 + \varepsilon/2) = \tfrac{1}{2} F(\varphi_1) + \tfrac{1}{2} F(\varphi_1 + \varepsilon) \tag{157}$$

Now assume that ε is small, and expand terms in a Taylor series through second-order terms in ε. This gives

$$F(\varphi_1) + \frac{\varepsilon}{2} \frac{\partial F}{\partial \varphi_1} + \frac{\varepsilon^2}{8} \frac{\partial^2 F}{\partial \varphi_1^2} = F(\varphi_1) + \frac{1}{2} \varepsilon \frac{\partial F}{\partial \varphi_1} + \frac{\varepsilon^2}{4} \frac{\partial^2 F}{\partial \varphi_1^2} \tag{158}$$

which implies that $\partial^2 F/\partial \varphi_1^2 = 0$. Thus, we conclude that

$$F = E\varphi_1 + L \tag{159}$$

for some constants E and L. Now, equation (152) gives us

$$E\varphi + L = \alpha_1(E\varphi_1 + L) + \alpha_2(E\varphi_2 + L) \tag{160}$$

which in turn gives

$$E\varphi = E(\alpha_1\varphi_1 + \alpha_2\varphi_2) \tag{161}$$

If $E \neq 0$, we conclude that $\varphi = \alpha_1\varphi_1 + \alpha_2\varphi_2 = \frac{1}{2}(\varphi_1 + \varphi_2)$ for all φ_1, φ_2. But this is unacceptable, since it contradicts (147). Therefore, we must have $E = 0$, which leaves us with $F = \text{const}$, which is then to be incorporated into the definition of the potential V.

Very briefly summarizing the discussion in this section, it has been shown that Γ [through terms of second order and degree; see (22)] has been reduced to the following form as a consequence of the requirement of superposition and bounded probability density:

$$\Gamma = \frac{1}{cr}\nabla^2 r \tag{162}$$

where $c < 0$.

Also, the relations $\varphi = \varphi(r_1, r_2, \varphi_1, \varphi_2)$ and $r = r(r_1, r_2, \varphi_1, \varphi_2)$ have been shown necessarily to be the well-known superposition relations of customary quantum theory.

3.6. Additions to Γ

Γ is the most general invariant (under coordinate rotations and inversions) through quantities of second order or degree in spatial differentiations that one could have. The question must now be considered as to whether this choice for Γ is too restrictive; i.e., can Γ be extended to include quantities of the third or higher order or degree? We shall find that the requirement of superposition prevents this from being the case. To illustrate what is involved, we consider the addition of the term

$$\rho \sum_1^3 \frac{\partial^3 \varphi}{(\partial x^j)^3}$$

to Γ, where $\rho = \rho(r, \varphi)$ is a scalar coefficient. This term is not an invariant (under coordinate inversion, for example) and therefore, technically, need not be considered; but it will be, since it is a very simple example of the point that is to be made. Inclusion of this term in Γ will produce an additional contribution to (23) described by the term

$$\rho\alpha_i^{(n)}\varphi_i^{(3-n)}C_n + \beta_i^{(n)}r_i^{(3-n)}C_n - \alpha_i\rho_i\frac{\partial^3\varphi_i}{(\partial x^j)^3} \tag{163}$$

where the summation on i, j is from 1 to 2; the summation over n is from 0 to 2; the superscripts in parentheses denote differentiations, so that

$\alpha_i^{(n)} \equiv \partial^n \alpha_i / \partial x^j \cdots \partial x^j$ (summed on j), and $\alpha_i^{(0)} \equiv \alpha_i$; and the C_n are constants given by

$$C_n = \begin{cases} 1 & \text{for } n = 0 \\ 2 & \text{for } n = 1 \\ 1 & \text{for } n = 2 \end{cases}$$

and $\rho_i \equiv \rho(r_i, \varphi_i)$.

Now, as a special case, choose all $r_i = \text{const}$, $\varphi_2 = \text{const}$, and $\partial\varphi_1/\partial x^1 = \text{const} \equiv \gamma_1$. Then the above contribution [i.e., (163)] becomes $\gamma^3\, \partial^2\alpha_1/\partial\varphi_1\, \partial\varphi_1$, or $\partial^2\alpha_1/\partial\varphi_1\,\partial\varphi_1 . (\partial\varphi_1/\partial x^1)^3$. Looking at (23) for this case, we get the relation

$$\mathscr{A}_{11}\left(\frac{\partial\varphi_1}{\partial x^1}\right)^2 + \frac{\partial^2\alpha_1}{\partial\varphi_1\,\partial\varphi_1}\left(\frac{\partial\varphi_1}{\partial x^1}\right)^3 = 0 \tag{164}$$

We see that the gradient factors do not cancel, since they are of different degrees, so we obtain the superposition equation

$$\mathscr{A}_{11} + \frac{\partial^2\alpha_1}{\partial\varphi_1\,\partial\varphi_1}\,\frac{\partial\varphi_1}{\partial x^1} = 0 \tag{165}$$

instead of the relation $\mathscr{A}_{11} = 0$ that we obtained earlier without the additional term in Γ. But we see that the above relation is not universal, since it depends explicitly on the spatial properties of φ_1.

From this example we see that whenever higher order or degree terms are added on to Γ, they will, for certain choices of the r_i and φ_i, lead to relations where not all the gradient terms in φ or r will cancel. This is merely because they are not all of the same degree (or order) as they were before the addition of other terms to Γ. In fact, for any choice of the r_i and φ_i that leaves some of the terms present from both second and higher order contributions one must end up with nonuniversal relations; and the presence of such relations contradicts the universality inherent in the notion of superposition.

Hence, the Γ discussed in this section 3 [equation (22)] is the most general one possible; i.e., is the most general one capable of being consistent with superposition.

4. DISCUSSION

The development in Section 3 was so lengthy that it should be helpful to summarize and point out the highlights of just what was accomplished there. In brief, the following development occurred. We assumed that the stochastic velocity was irrotational, describing quantum phenomena by a dynamic equation of the form (where **B** depends on the state of the system, but not on its time derivative)

$$m\frac{d\mathbf{v}}{dt}=\mathbf{B} \tag{166}$$

together with the continuity equation. Even though the above relation seems most reasonable, we note that it is an *assumption*; it need not be so. For example, we could conceivably have a relation of the form $m\,d^2\mathbf{v}/dt^2=\mathbf{C}$, where **C** depends on the system state (and not on its time derivative). Or we might have some nonlocal integral form involving **v** on the left-hand side of the relation instead. However, it would be difficult imagining how such descriptions could yield an average behavior characterized by Newton's law of inertia. Next, assuming that **v** is irrotational, we derived the relation

$$\frac{(\nabla\varphi)^2}{2m}+V+\Gamma=-\varphi_{,t} \tag{167}$$

where we momentarily assumed a certain restricted form for Γ. Now, the requirement of superposition together with the requirements that r^2 should be bounded led to the result that $\Gamma=\nabla^2 r/cr$, where $c<0$. It was then shown that this form for Γ was generally valid if superposition were to hold. In effect, then, the few assumptions used, together with that of superposition, were shown to imply the relations

$$\frac{(\nabla\varphi)^2}{2m}+V+\frac{1}{cr}\nabla^2 r=-\varphi_{,t} \tag{168}$$

and

$$\nabla\cdot(r^2\,\nabla\varphi)+mr^2_{,t}=0 \tag{169}$$

even when the above restriction on Γ was dropped. However, letting $c^{-1}=-\hbar^2/2m$, these two equations are seen to be equivalent to the single equation

$$\left(-\frac{\hbar^2}{2m}\nabla^2+V\right)\Psi=i\hbar\,\partial_t\Psi \tag{170}$$

where $\Psi\equiv r\exp[(i/\hbar)\varphi]$. Thus, the Schrödinger equation is recovered.

Finally, it was also shown that the relations $\varphi=\varphi(r_1, r_2, \varphi_1, \varphi_2)$ and $r=r(r_1, r_2, \varphi_1, \varphi_2)$ are none other than the well-known superposition relations of ordinary quantum mechanics.

In the final section, we will show that the assumption that **v** is irrotational need not be made in the above development, since we make a compelling argument to prove that rotational **v** cannot satisfy the superposition principle. Thus, the conclusion, equation (170), follows without the proviso that $\mathbf{v}$ be irrotational. The Schrödinger equation will then have been founded on the assumptions that $d\mathbf{v}/dt =$ function of state, where $\mathbf{v}$ is nonuniquely *defined* by the continuity relation, r^2 is bounded, and the superposition principle holds.

5. GENERALIZED QUANTUM MECHANICS

Up to this point we have given a compelling demonstration that the only quantum theory—with an irrotational velocity field—consistent with superposition and a few other stipulations is the conventional Schrödinger quantum description. We shall extend this conclusion to cover the case where the stochastic velocity need not be irrotational.

Again, we take the dynamic equation to be

$$m\frac{d\mathbf{v}}{dt} = \mathbf{B} \tag{171}$$

where

$$\mathbf{v} = \frac{1}{m}\nabla\varphi + \mathbf{A} \tag{172}$$

is the (now assumed rotational) stochastic velocity field, which is again (nonuniquely) defined by the continuity equation

$$\nabla \cdot (r^2\mathbf{v}) + r^2_{,t} = 0 \tag{173}$$

Here, φ is some unspecified scalar function, and $\mathbf{A}$ is an unspecified solenoidal vector field. Again, $\mathbf{B}$ is a vector field somehow depending on the system state (i.e., on r, φ, $\mathbf{A}$), but not on its time derivative. Inserting the expression (172) into the two equations above then gives the relations governing the description as

$$\nabla\left[\frac{1}{2m}(\nabla\varphi)^2 + \partial_t\varphi\right] + (\nabla\varphi\cdot\nabla)\mathbf{A} + (\mathbf{A}\cdot\nabla)\nabla\varphi + m(\mathbf{A}\cdot\nabla)\mathbf{A} + m\,\partial_t\mathbf{A} = \mathbf{B} \tag{174}$$

and

$$\frac{r}{2m}\nabla^2\varphi + \frac{1}{m}\nabla r\cdot\nabla\varphi + r_{,t} + \mathbf{A}\cdot\nabla r = 0 \tag{175}$$

Now, let us investigate the consequence of the superposition principle holding for the continuity equation (175) in a simple case: We assume that the component solutions are such that $\nabla r_\alpha = 0 = \nabla \varphi_\alpha$ for $\alpha = 1, 2$. For the superposed state $r, \varphi, \mathbf{A}$ we have in general

$$\varphi = \varphi(r_1, r_2, \varphi_1, \varphi_2, \mathbf{A}^{(1)}, \mathbf{A}^{(2)})$$
$$r = r(r_1, r_2, \varphi_1, \varphi_2, \mathbf{A}^{(1)}, \mathbf{A}^{(2)})$$
$$\mathbf{A} = \mathbf{A}(r_1, r_2, \varphi_1, \varphi_2, \mathbf{A}^{(1)}, \mathbf{A}^{(2)})$$

since the vector fields $\mathbf{A}$, $\mathbf{A}^{(\alpha)}$ now form part of the description of the state of the system. (Note that we shall use Greek indices going from 1 to 2 to indicate the component systems and Latin indices going from 1 to 3 to indicate vector components.) Further, we write

$$\nabla \varphi = \frac{\partial \varphi}{\partial A_i^{(\alpha)}} \nabla A_i^{(\alpha)} \equiv \mathcal{M}_{i\alpha} \nabla A_i^{(\alpha)} \tag{176}$$

where no other terms enter here on the right-hand side because we are choosing $\nabla r_\alpha = 0 = \nabla \varphi_\alpha$ for $\alpha = 1, 2$. Also, the summation convention, on all indices, Greek and Latin, is used throughout. We also have

$$\nabla r = \frac{\partial r}{\partial A_i^{(\alpha)}} \nabla A_i^{(\alpha)} \equiv \mathcal{N}_{i\alpha} \nabla A_i^{(\alpha)} \tag{177}$$

and we also have expressions similar to the above for $\varphi_{,t}$ and $r_{,t}$, respectively. Putting these expressions into (175) gives

$$\nabla A_i^{(\alpha)} \cdot \nabla A_j^{(\beta)} \left(\frac{r}{2m} \frac{\partial \mathcal{M}_{i\alpha}}{\partial A_j^{(\beta)}} + \frac{1}{m} \mathcal{M}_{i\alpha} \mathcal{N}_{j\beta} \right) + \frac{r}{2m} \mathcal{M}_{i\alpha} \nabla^2 A_i^{(\alpha)}$$
$$+ \mathcal{N}_{i\alpha} \nabla A_i^{(\alpha)} \cdot \mathbf{A} + \mathcal{N}_{i\alpha} A_{i,t}^{(\alpha)} + \frac{\partial r}{\partial \varphi_\alpha} \varphi_{\alpha,t} + \frac{\partial r}{\partial r_\alpha} r_{\alpha,t} = 0 \tag{178}$$

As in the case where $\mathbf{v}$ was purely irrotational, the next thing we would like to do here is to reexpress the time derivatives above in terms of spatial derivatives by using the original equations (174) and (175) for the component solutions [see the discussion before equation (8)]. But in the present case this willl lead to difficulty, as we now discuss. In order to express the $\mathbf{A}_{,t}^{(\alpha)}$ and $\varphi_{(\alpha),t}$ in terms of spatial quantities, we proceed as follows. The dynamic equation (174) governing each component solution has the form $\partial_t \nabla \varphi_\alpha + m\, \partial_t \mathbf{A}^{(\alpha)} = \mathbf{b}^{(\alpha)}$ for some vector field $\mathbf{b}$ depending only on spatial derivatives. Taking divergences, and remembering that $\mathbf{A}^{(\alpha)}$ is solenoidal, we have $\nabla^2 \partial_t \varphi_\alpha = \nabla \cdot \mathbf{b}^{(\alpha)}$, which then yields the relation

$$\partial_t \varphi_\alpha = -\frac{1}{4\pi} \int \frac{\nabla \cdot \mathbf{b}^{(\alpha)}}{|\mathbf{x}' - \mathbf{x}|} d\mathbf{x}' \tag{179}$$

Again, taking the curl of the above expression, we obtain $m\, \partial_t \nabla \times \mathbf{A}^{(\alpha)} = \nabla \times \mathbf{b}^{(\alpha)}$. Now setting $\mathbf{A}^{(\alpha)} = \nabla \times \mathbf{S}^{(\alpha)}$ for some $\mathbf{S}^{(\alpha)}$ and further gauging $\mathbf{S}^{(\alpha)}$

so that $\nabla \cdot \mathbf{S}^{(\alpha)} = 0$, we obtain $-m\, \partial_t \nabla^2 \mathbf{S}^{(\alpha)} = \nabla \times \mathbf{b}^{(\alpha)}$, which finally implies that

$$\partial_t \mathbf{A}^{(\alpha)} = \frac{1}{4\pi m} \nabla \times \int \frac{\nabla \times \mathbf{b}^{(\alpha)}}{|\mathbf{x}' - \mathbf{x}|} \, d\mathbf{x}' \tag{180}$$

We see that the relations connecting $\varphi_{\alpha,t}$ and $\partial_t \mathbf{A}$ to spatial derivatives are nonlocal. When these nonlocal expressions are substituted into (178) for the $\mathbf{A}^{(\alpha)}_{,t}$ and $\varphi_{\alpha,t}(r_{\alpha,t} = 0$ in our special case), we end up with a relation involving terms with the usual factors, $\nabla^2 A_i^{(\alpha)}, \nabla A_i^{(\alpha)} \cdot \nabla A_j^{(\beta)}, \ldots$, as well as the above nonlocal expressions. However, there is generally no way that these local and nonlocal expressions can cancel each other to yield the universal relations demanded by superposition. It is conceivable that certain functional choices for the $\mathbf{A}^{(\alpha)}$ might allow a partial cancellation with the result depending on the particular functional form chosen for the $\mathbf{A}^{(\alpha)}$. However, this is unsatisfactory, as we noted before. Or, in certain circumstances, these nonlocal expressions might, in fact, reduce to local ones. For example, if $\mathbf{b}^{(\alpha)} = \nabla \chi^{(\alpha)}$ for some scalar function $\chi^{(\alpha)}$, then we have that

$$\partial_t \varphi_\alpha = -\frac{1}{4\pi} \int \frac{\nabla^2 \chi^{(\alpha)}}{|\mathbf{x}' - \mathbf{x}|} \, d\mathbf{x}' = \chi^{(\alpha)} \tag{181}$$

In fact, this is precisely the way we would recapture the $\mathbf{A} \equiv 0$ description (already discussed in Section 3) from the present one. However, $\mathbf{b}^{(\alpha)} = \nabla \chi^{(\alpha)}$ implies that $\partial_t \mathbf{A}^{(\alpha)} = 0$, according to (180), so the $\mathbf{A}^{(\alpha)}$ and $\mathbf{A}$ would necessarily be constant in time. But if the $\mathbf{A}^{(\alpha)}$ are not spatially constant as well, then the $\mathbf{b}^{(\alpha)}$ are seen to be not *identically* given as the gradient of any function of the system state, so the relation $\mathbf{b}^{(\alpha)} = \nabla \psi^{(\alpha)}$ becomes another *condition* that must be satisfied. This means that we have five unknowns (r, φ, and three components of $\mathbf{A}$) and eight equations, consisting of the continuity relation, $\nabla \cdot \mathbf{A} \equiv 0$, $m\, d\mathbf{v}/dt = \mathbf{B}$, and $\mathbf{b} = \nabla \chi$; and this is unsatisfactory. Finally, if the $\mathbf{A}$, $\mathbf{A}^{(\alpha)}$ *are* constant (in space and time), $\mathbf{v}$ is again irrotational. We therefore see that the assumption $\mathbf{A} \neq$ const leads to grave problems if superposition is to hold. We therefore conclude that $\mathbf{A} \equiv 0$ ($\mathbf{A} =$ const. is equivalent to $\mathbf{A} \equiv 0$ with another choice of φ) and we see then that only an irrotational $\mathbf{v}$ can reasonably be compatible with the superposition principle.

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